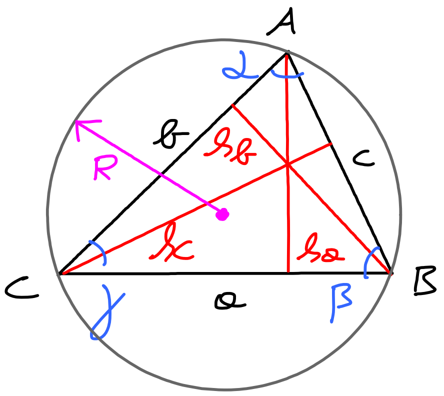
Problem 11951 - 01 - O. Kouba (Syria).

Let ABC be a triangle that is not obtuse. Denote by a, b, and c the lengths of the sides opposite A, B, and C, respectively, and denote by h_a, h_b, and h_c the lengths of the altitudes dropped from A, B, and C, respectively. Prove that

$$\frac{a^2}{h_b^2+h_c^2}+\frac{b^2}{h_c^2+h_a^2}+\frac{c^2}{h_a^2+h_b^2}<\frac{5}{2}.$$

Show also that $5/2$ is the smallest possible constant in this inequality.

SOLUTION PROPOSED BY GIUSEPPE MUSINU FROM RAVENNA (ITALY)
17/03/17



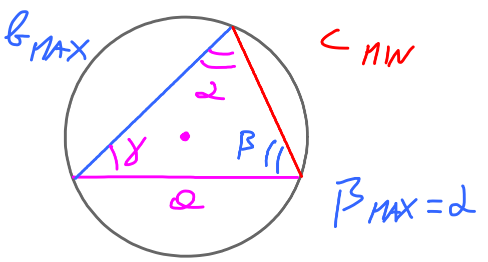
$$\begin{cases} a = 2R\sin\alpha \\ b = 2R\sin\beta \\ c = 2R\sin\gamma \end{cases} \qquad \begin{cases} h_a^2 = b^2\sin^2\gamma = c^2\sin^2\beta \\ h_b^2 = a^2\sin^2\gamma = c^2\sin^2\alpha \\ h_c^2 = a^2\sin^2\beta = b^2\sin^2\alpha \end{cases}$$

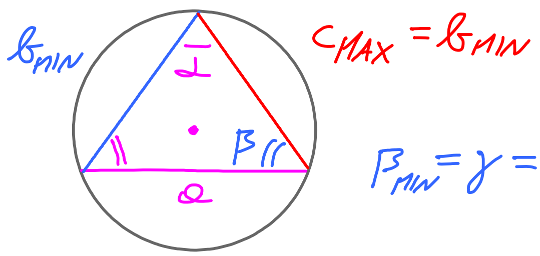
$$\frac{a^2}{h_b^2+h_c^2}+\frac{b^2}{h_a^2+h_c^2}+\frac{c^2}{h_a^2+h_b^2}=$$

$$\frac{1}{\sin^2\alpha+\sin^2\beta}+\frac{1}{\sin^2\alpha+\sin^2\gamma}+\frac{1}{\sin^2\beta+\sin^2\gamma} \quad (1)$$

LET WLOG $\alpha \geq \beta \geq \gamma$

$\overset{\triangle}{ABC}$ IS NOT OBTUSE $\Rightarrow$ $$\begin{cases} \frac{\pi}{3} \leq \alpha \leq \frac{\pi}{2} \\ \frac{\pi}{2}-\frac{\alpha}{2} \leq \beta \leq \alpha \\ \gamma = \pi-\alpha-\beta \end{cases}$$





LEMMA $\boxed{\sin^2\alpha+\sin^2\beta+\sin^2\gamma \geq 2}$ (2)

PROOF $\sin^2\alpha+\sin^2\beta+\sin^2\gamma = \dfrac{1-\cos 2\alpha}{2} + \dots =$

$$= \frac{3}{2} - \frac{1}{2}\left(\cos 2\alpha + \cos 2\beta + \cos 2\gamma\right) = *$$

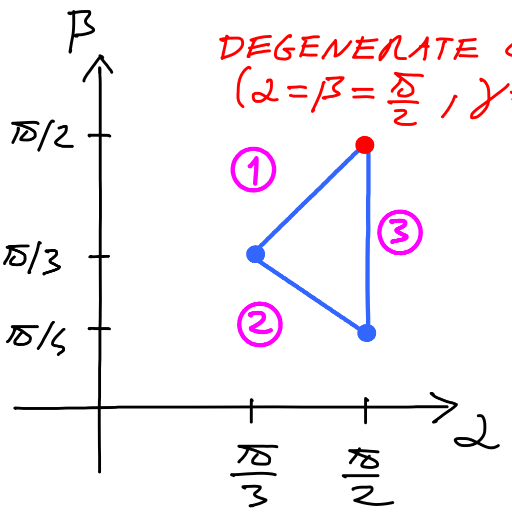
$$(3)\ \begin{cases} \cos 2\alpha = \cos(2\pi - 2(\beta+\gamma)) = \cos(2(\beta+\gamma)) = \\ \qquad = 2\cos^2(\beta+\gamma) - 1 \\ \cos 2\beta + \cos 2\gamma = 2\cos(\beta+\gamma)\cos(\beta-\gamma) \end{cases}$$

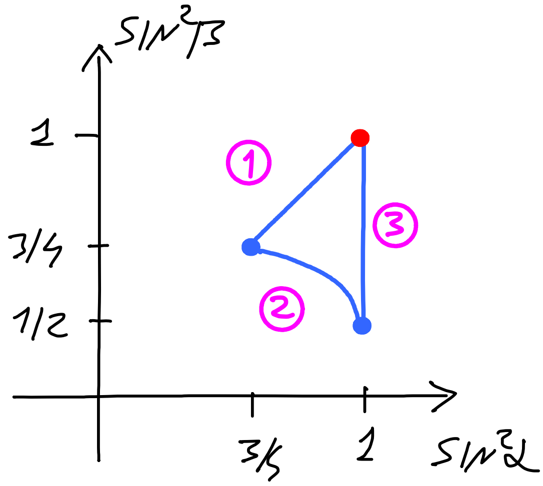
$$(4)\ \begin{cases} \cos 2\alpha + \cos 2\beta + \cos 2\gamma \overset{(3)}{=} 2\cos(\beta+\gamma)[\cos(\beta+\gamma)+\cos(\beta-\gamma)] - 1 = \\ = 4\cos(\beta+\gamma)\cos\beta\cos\gamma - 1 \end{cases}$$

$$* = 2 - 2\cos(\beta+\gamma)\cos\beta\cos\gamma$$

$$\Rightarrow \sin^2\alpha+\sin^2\beta+\sin^2\gamma \begin{cases} > 2 & \alpha+\beta > \frac{\pi}{2} & \text{"ACUTE"} \\ = 2 & \alpha+\beta = \frac{\pi}{2} & \text{"RIGHT"} \\ < 2 & \alpha+\beta < \frac{\pi}{2} & \text{"OBTUSE"} \end{cases} \qquad \square$$

$$\Rightarrow (1) \leq \frac{1}{\sin^2\alpha+\sin^2\beta} + \frac{1}{2-\sin^2\alpha} + \frac{1}{2-\sin^2\beta} \qquad (5)$$





① "ISOCELES" $\alpha=\beta$

② "ISOCELES" $\beta=\gamma=\frac{\pi}{2}-\frac{\alpha}{2} \Rightarrow \sin^2\beta = \cos^2\frac{\alpha}{2} = \dfrac{1+\sqrt{1-\sin^2\alpha}}{2}$

③ "RIGHT" $\alpha=\frac{\pi}{2}$

LET $\sin^2\alpha = x \quad \sin^2\beta = y$

$$(5) = f(x,y) = \frac{1}{x+y} + \frac{1}{2-x} + \frac{1}{2-y}$$

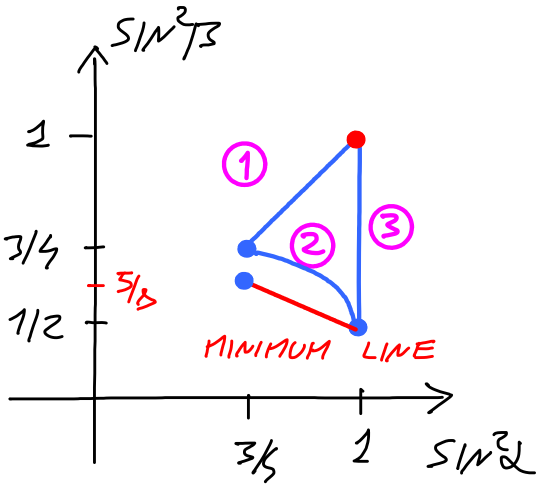
$$f_y = \frac{-4-\cancel{y^2}+4y+\cancel{y^2}+x^2+2xy}{(x+y)^2(2-y)^2} = 0$$

$$x^2+2xy+4y-4=0 \qquad 2y(\cancel{2+x}) = (\cancel{2+x})(2-x) \; {\scriptstyle x \neq -2} \qquad y = \frac{2-x}{2}$$

$$x^2+2xy+4y-4 = x(\overbrace{x+2y}^{\geq 2})+4y-4 \overset{\text{LEMMA}}{\geq} x\cdot 2+4y-4 =$$

$$= 2(\overbrace{x+2y}^{\geq 2})-4 \geq 2\cdot 2-4 = 0$$

$\Rightarrow g(y) = f(\bar{x},y)$ IS AN INCREASING FUNCTION WITH A LOCAL MINIMUM FOR $y = \frac{2-\bar{x}}{2} \leq \frac{1+\sqrt{1-\bar{x}^2}}{2}$



$\Rightarrow$ MAX $g(y) \in$ 1 $\quad x = y$

$$s(y) = f(y,y) = \frac{1}{2y} + \frac{2}{2-y} \qquad s'(y) = -\frac{1}{2y^2} + \frac{2}{(2-y)^2} = 0$$

$$-4-y^2+4y+4y^2=0 \qquad 3y^2+4y-4=0 \qquad y = \frac{-2\pm\sqrt{16}}{3} = \begin{cases} 2/3 \\ -2 \end{cases}$$

$$3y^2+4y-4 = y(\overbrace{3y}^{\geq 2})+\overbrace{3y}^{\geq 2}+y-4 \geq 2y+2+y-4 =$$

$$= \overbrace{3y}^{\geq 2}-2 \geq 2-2 = 0 \Rightarrow s'(y) \geq 0 \quad s'=0 \iff y = 2/3$$

$\Rightarrow$ MAX $g(y) = f(1,1) = \frac{1}{2} + 1 + 1 = \frac{5}{2}$

$\Rightarrow (1) \leq (5) \leq \frac{5}{2} \qquad \Rightarrow (1) \leq \frac{5}{2}$

AND THE "=" HOLDS FOR THE DEGENERATE

CASE $(\alpha = \beta = \frac{\pi}{2}, \gamma = 0)$, THUS FOR

$\overset{\triangle}{ABC}$ NOT OBTUSE: $(1) < \frac{5}{2}$ $\square$